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Structural information theory

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A method of defining structural information in Hilbert space is described. Instead of commencing an info theory analysis with a simple signal and progressing to an n-dimensional information space, the procedure can be followed: the signal is defined completely in both real and imaginary spaces, and represented as a dimensionless number in Hilbert space. Thus, a signal can be analyzed into a Hilbert measure of information. This measure has similarities to the relativity of states of Everett's (1957) reformu of quantum mechanics involving observer–observed interaction. Whereas Everett's theory deals with totality of all possible ways in which a state function can be decomposed into the sum of products of functions for subsystems of the overall system, the theory presented here deals with the totality of possible in which a Hilbert space measure of signal state can be decomposed by signal definition into reciprocally measures of the state for subsystems of the overall system. In the overall system presented here, a signal h a concise representation either as a quadratic form on a Hilbert space, or as a Fourier coefficients in function space. The total information content of a signal qualified as a dimensionless number in Hilbert may be given a representation in familiar units by the use of such methods. A primary postulate of qu mechanics is obeyed in that all physically relevant information about a system is derivable from the know of the state function of the system. Thus, a signal is defined *firstly* as an information measure, *secondly* in the familiar units of frequency, bandwidth, midperiod, and duration.

Subject Classification: 15.2

INTRODUCTION

The *logon*—an elementary signal or structural "quantum" of information measurement—is defined (Stewart, 1931; Kock, 1935; Gabor, 1946; Kharkevich, 1960; Brillouin, 1962; Pimonow, 1962; Barrett, 1971) as

$$\Delta f \cdot \Delta t = \tfrac{1}{2}, \tag{1}$$

where Δf is the "effective" signal bandwidth and Δt is the "effective" signal duration. Such a unit can define certain aspects of the oscillatory response of any system of one mechanical or structural degree of freedom, with a second degree of freedom given by the duration of its response considered as a signal. One illustration is that a reed responding to an electrical signal has one mechanical degree of freedom with a response defined by solutions of the homogenous equation (Corliss, 1963):

$$M\ddot{x} + D\dot{x} + kx = 0, \tag{2}$$

where M is inertia, D dissipation, k the coefficient of restitution, and x the mechanical coordinate with one degree of freedom.

More complex systems present problems in relating the elementary signal to the structural system considered, as more structural degrees of freedom are involved. For example, the central nervous system (CNS), viewed as an information conveying system, has more than one degree of freedom. The sensory receptor of the auditory system behaves with continuously varying coefficients (equivalent to M, D, and k in Eq. 2), and has at least four degrees of freedom (Rink, 1970). Presently, there is an abse able mathematical treatment of the b fluid or endolymph within the cochlear p Békésy, 1965). This paper will treat only of freedom case, but another communica 1972b) points out some interesting implic more than one degree of freedom is specific mathematics required to under mechanics will not be provided here, bu theory with which such a system must

Before commencing a complete signal information theory terms, it is first nec phasize that the information theory add concerned with information structure and Shannon theory, which is a related but (Barrett, 1972d). Secondly, the defining terms of bandwidth, duration, midfr midperiod requires four distinct units o and not two (Barrett, 1972c). The statement are as follows.

The definition of an information meas only two parameters—signal bandwidth casts doubt on the elegance of the usual signal by circular functions in the tim instance, if a signal is defined by a circ together with statements concerning and bandwidth—one is confined by which is adequate enough for most purp methods for signal redefinition in Other writers (cf. Gabor, 1946) have p the Fourier methods require infinite logon definition has shortcomings, as [illegible] relation, but does not reference in one [illegible]nition: (a) the center frequency of the [illegible] measured in cycles/second or hertz); (b) the [illegible] of the signal (measured in seconds/cycle), or [illegible] the variance of energy dispersion in the [illegible] domain is equivalent or different from that [illegible] domain. Thus, neither circular functions nor [illegible] offer a complete definition of the signal. [illegible] of functions does not define the concept of a [illegible] merely expresses the values of a signal in [illegible] domains. For signal definition, a set of functions [illegible] basis, but coefficients are necessary in vector [illegible]

[illegible] argument against the conventional use of [illegible] functions is that Fourier's theorem treats time [illegible] (Gabor, 1946, 1947a, 1947b, 1953). The use [illegible] functions appears based on, for our purposes, [illegible] pragmatism. Their usefulness lies in the [illegible] of defining a signal in circular function form [illegible] referent to the frequency domain; i.e., [illegible] time is presupposed, a circular function has an [illegible] definition in the frequency domain. It is common [illegible] that signal redefinition may be accomplished [illegible] integral transform. However, the presupposition [illegible] time is questionable and circular functions [illegible] be too restrictive a language for structural [illegible] theory.

[illegible]structive to consider the nature of integral [illegible]—especially the nature of the exponential [illegible]. The exponential kernel may be defined as the [illegible] the important functional equation (Cauchy, [illegible] 1959, p. 139):

$$F(\epsilon_1+\epsilon_2)=F(\epsilon_1)F(\epsilon_2), \tag{3}$$

[illegible] the solution to this equation and ϵ is complex [illegible] positive real part. As a kernel of an [illegible] exponential provides a means for rede[illegible] in any domain defined by the exponent [illegible] and the integral operation. This is [illegible] solution to Eq. 3 is a function of a peculiar [illegible] a function that defines the addition of a [illegible] the multiplication of a function as equiva[illegible]cation by the exponential function used [illegible] an integral transform provides an entrance [illegible] domain, which is referenced by the variable [illegible] of the exponential function, but which [illegible] in the integration space. Geometri[illegible]formation is performed on the signal, [illegible] signal has a *referent to the second domain or* [illegible] functions do have a referent to time and [illegible], but an integral transform will be [illegible] function defined only over one domain. [illegible] of integral transform methods—[illegible] mathematical correctness exists—may, [illegible] a physically impossible signal [illegible]

[illegible]form method is one of a number [illegible] signal redefinition. Because circular functions are popularly used for signal definition, and are assumed to be of infinite duration, the most common domain in which to redefine a signal is the frequency domain using the Fourier transform method. Since information about the duration of the signal is required for logon representation, a structural information theory is, perhaps, more easily approached with other methods.

More concise methods involve a complete signal definition that refers to four observables: the natural or midperiod of the signal (t_0), the duration of the signal (Δt), the center or midfrequency of the signal (f_0), and the signal bandwidth (Δf). Quantification of the signal in these dimensions involves different units: seconds/cycle for the midperiod (t_0), seconds for the duration (Δt), cycles/second or hertz for the midfrequency (f_0), and cycles for the bandwidth (Δf) (Barrett, 1972c). The midperiod (t_0) and the midfrequency (f_0) are defined over both the frequency and time domains and are reciprocal measures of rate or succession involving *sequence* spaces in the terms of functional analysis. The duration (Δt) and the bandwidth (Δf) are defined separately, do *not* measure rate or succession, and involve *integration* spaces (Maddox, 1970).

It may be necessary to defend the usage of different units for signal bandwidth and midfrequency. Bandwidth has no real representation in the time domain—according to our analysis. Bandwidth, a number, refers only to the frequency domain. Association with circular functions, which do refer to both the frequency and time domains, may have obscured the singular nature of signal bandwidth. The use of Fourier methods introduces a circular function redefinition in the frequency domain in which signal bandwidth may also be defined, and thereby may conceal the fact that no temporal referent exists for a bandwidth variable.

An example may be of use: a bandwidth of, say, 20 cycles, is not a rate, i.e., the units should not be hertz, as the upper and lower bounds of the bandwidth may refer to 180 and 200 hertz or 1180–1200 hertz, or any other numbers with a difference of 20. It is, therefore, necessary to know the average or midfrequency (f_0) of the signal *before* the upper and lower bounds of the bandwidth (measured in hertz) can be given a precise significance. According to this conception, bandwidth is only indirectly equal to the upper signal frequency passed minus the lower signal frequency passed. It may be overemphasizing to point out that the result of such a subtraction is not measured, therefore, in hertz. Bandwidth quantifies a physical occurrence that is neither a rate nor a speed.

I. ARGUMENT

From this separation of units defining a signal, our goal of a more concise, if not "truer," signal analysis may be pursued, to account for the disparate nature of information defined over an integration space and that

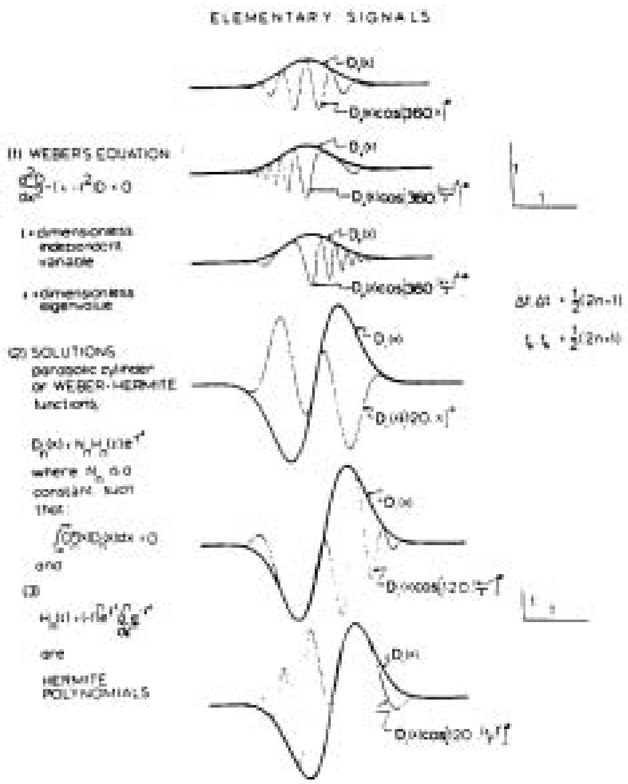

FIG. 1. The elementary signal defined by the conditions—$\Delta f \cdot \Delta t = \frac{1}{2}$ and $f_0 \cdot t_0 = \frac{1}{2}$—$D_0(x)$ in the notation of this figure—is the first in a series of modulations and may also modulate different functions. Here are shown elementary signals with amplitude modulations $D_0(x)$ and $D_1(x)$. In the first of each three the modulated signal is a sinusoidal function; in the second and third the amplitude modulated signal is also frequency modulated—either ascending or descending. The frequency modulation is such that $\Delta f \cdot f_0 = \frac{1}{2}(2n+1)$ where Δf is change in frequency and f_0 is center frequency. [From Barrett, 1971, p. 133.]

defined over a sequence space. Reconciliation of the two spaces is provided by a Hilbert space representation.

The requirements for Eq. 1 may be obtained in the time and frequency domains by the following definitions (Gabor, 1947):

$$s(t) = e^{-c^2(t-t_0)^2} \cdot e^{i2\pi f_0 t}; \quad s(f) = e^{-(\pi/c)^2(f-f_0)^2} \cdot e^{-i2\pi t_0 f}. \tag{4}$$

The c in Eq. 4 is related to the duration and the bandwidth of the signal:

$$\Delta t = \sqrt{\frac{\pi}{2}} \Big/ c, \quad \Delta f = c/\sqrt{2\pi} \tag{5}$$

The expressions $e^{-c^2(t-t_0)^2}$ and $e^{-(\pi/c)^2(f-f_0)^2}$ refer to the spread or dispersion of signal energy in the time and frequency domains. But Δt and Δf are only obliquely referenced by these expressions, because Δt does not equal $(t-t_0)^2$ and Δf does not equal $(f-f_0)^2$. This is because—to reiterate our point—f, t, f_0, and t_0 are variables of a sequence space measuring rates, and Δf and Δt are variables of an integration space *not* measuring rates. The association[1] of $(t-t_0)^2$ [illegible] $(f-f_0)^2$ and Δf arises because a [illegible] seconds/cycle can be related to a [illegible] seconds, and, likewise, a measurement in [illegible] (hertz) can be related to a measurement in [illegible] *without direct equalization.*

Given four observables of a signal instead of [illegible] reorganizing of informational quanta is called [illegible] Another bounding condition for an elementary [illegible] may be defined relating the logical resolution [illegible] f_0 and t_0:

$$f_0 \cdot t_0 = \tfrac{1}{2}.$$

In effect, Eq. 6 amounts to setting the constant [illegible] Eq. 4 equal to unity, and an elementary signal [illegible] a four-dimensional measure in both integration [illegible] sequence spaces. Thereby, the variety of elementary [illegible] logon signal forms obtainable by the Gabor [illegible] formulation due to variation in the value of the constant c no longer exists once information is defined [illegible]

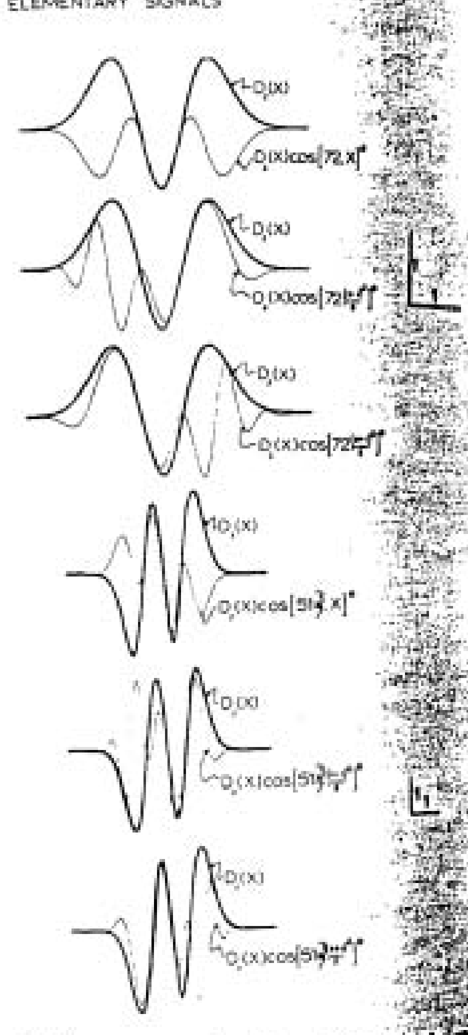

FIG. 2. Elementary signals with amplitude modulation [illegible] and $D_2(x)$; otherwise as in Fig. 1. [From Barrett, 1971, [illegible]

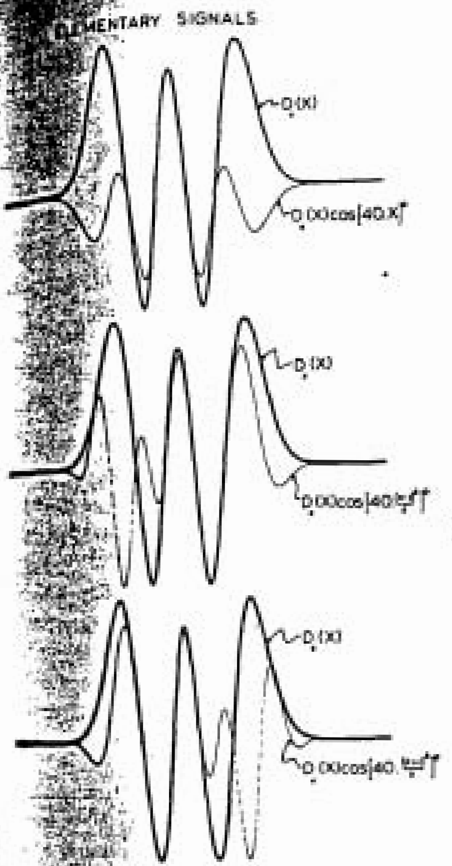

Fig. 3. Elementary signals with amplitude modulation $D_4(x)$; otherwise as in Fig. 1. [From Barrett, 1971, p. 134.]

in integration space but also sequence spaces. A four-dimensional informational quantum has a rigidly defined form for every minimum condition, such as those of Eqs. 1 and 6 (cf. Figs. 1–4).

The definition of elementary signals in vector form not only expresses values of a signal but provides an compact definition. Elementary signals can be defined (Barrett, 1972a):

$$\begin{aligned}\alpha &= \Delta f\cdot\Delta t + jf_0\cdot t_0 \text{ in subspace } \mathfrak{M}_1,\\ \beta &= \Delta f\cdot\Delta t - jf_0\cdot t_0 \text{ in subspace } \mathfrak{M}_2.\end{aligned}\tag{7}$$

In this formulation, two pairs of canonically conjugate variables (Courant and Hilbert, 1953, p. 223) are referenced, with

$$\beta=\alpha^{*},\tag{8}$$

where * denotes complex conjugation. A signal is defined as the inner product $\alpha\alpha^{*}$ of two vectors **x** and **y**, with contravariant components in the subspaces $\mathfrak{M}_1$ and $\mathfrak{M}_2$:

$$\begin{aligned}\mathbf{x} &= \mathbf{i}\Delta f\cdot\Delta t + jf_0\cdot t_0 \text{ for } \mathfrak{M}_1,\\ \mathbf{y} &= \mathbf{i}\Delta f\cdot\Delta t - jf_0\cdot t_0 \text{ for } \mathfrak{M}_2,\end{aligned}\tag{9}$$

where **i** and **j** are orthogonal unit vectors; *i* has the dimension of cycles-seconds and *j* (to a stationary observer) is dimensionless.[3] An inner product or Hilbert space $\mathcal{H}$ may be defined as

$$\mathcal{H}=\mathfrak{M}_1\cdot\mathfrak{M}_2.\tag{10}$$

The elementary signal definitions of Eq. 7 thus reduce to a real number. With $\Delta f\cdot\Delta t=\frac{1}{2}$ and $f_0\cdot t_0=\frac{1}{2}$, $\alpha\alpha^{*}=\frac{1}{2}$. If an arbitrary signal is redefined as a set of elementary signals, then $\alpha\alpha^{*}$ is a real number measure of the signal's information structure.

The procedure used in obtaining this simple signal definition is the reverse of that normally used in information theory analyses. Usually, a signal is defined as a simple circular function and the derivation proceeds until a complex information space definition results. Here, a complex signal is defined (exactly), and a simple information structure definition is obtained.

II. PRELIMINARY DISCUSSION

There are certain similarities in the procedure used here to that of the "relative state" formulation of quan-

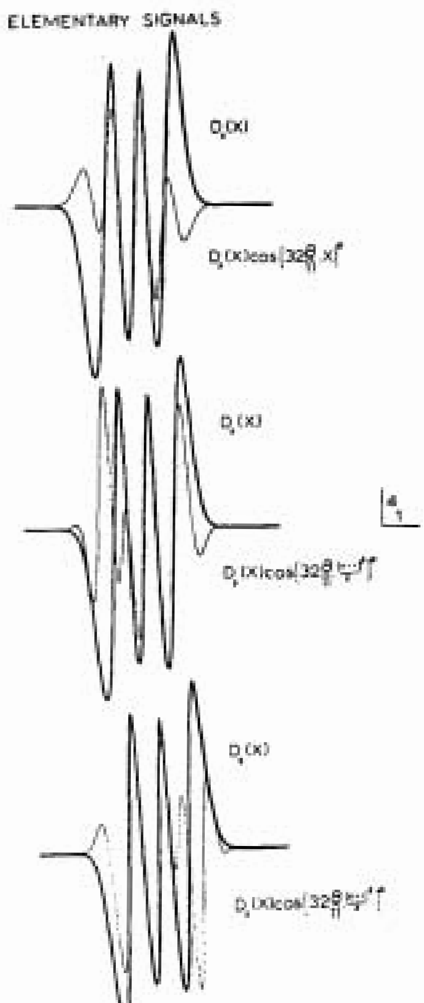

Fig. 4. Elementary signals with amplitude modulation $D_5(x)$; otherwise as in Fig. 1. [From Barrett, 1971, p. 134.]

tum mechanics (Everett, 1957; Wheeler, 1957; DeWitt, 1968), with the difference that the topic addressed here is that of "relative measure" rather than a "relative state." However, it is instructive to notice the similarities.

Everett (1957) proposed a physical system completely describable by a state function ψ, which is an element of Hilbert space and gives information specifying the probabilities of the results of various observations that can be made on the system by "external observers." Everett's concern with the interaction of system and observer is not a primary concern of this paper, and I have also addressed the topic of the results of these observations rather than the means of obtaining them. Everett thus addresses the problem of observer–observed interaction, but the underlying physical processes of his solution are similar.

Our short exposition of Everett's theory continues: a "relativity of states" is postulated, namely, that with any arbitrary chosen state for one subsystem, there will correspond a unique *relative state* for the remainder of the composite system; the states occupied by the subsystems are not independent but *correlated*. Similarly, I have defined signals in subdomains ($\mathfrak{M}_i$) linked by the reciprocal relations of Eqs. 1 and 6. A signal defined in one subdomain is a "relative measure" for the other subdomain.

> In the new or "relative state" formalism, this model associates with an isolated system a state function that obeys a linear wave equation. The theory deals with the totality of all possible ways in which this state function can be decomposed into the sum of products of state functions for subsystems of the overall system—and nothing more [Wheeler, 1957, p. 463].

This paper, on the other hand, addresses the problem of the totality of possible ways in which a measure of state defined by a Hilbert space representation can be decomposed by signal definition into reciprocally related measures, thus defining the state for subsystems of the overall system.

There is a need to conceptualize a "relative measure" definition of information in the CNS. Information is registered in the CNS in an analog fashion and reflects the subdomain definitions presented here. The frequency domain is physically represented at the basilar membrane and other stations of the auditory pathway, where a "tonotopic" arrangement of nerve fibers exists. The time domain is in evidence in the synchronous firing of auditory cells. Taken as a volume, the four observables of information measurement are apparent in the CNS related to the four dimensions used in defining the vectors **x** and *y*. The rotations or redefinitions of axes obtained at the various stations in the CNS can be considered as follows: let a vector **X**, which is not an elementary signal, be defined in elementary signal terms by its expansion of coefficients C_i, where $C_1=\Delta F\cdot\Delta T$ and $C_2=F_0\cdot T_0$ (ΔF, ΔT, F_0, and T_0 are an arbitrary signal's bandwidth, duration, midfrequency, and midperiod, respectively). Let a [illegible] not an elementary signal, be defined by [illegible] coefficients, D_i, where $D_1=\Delta F\cdot\Delta T$ and [illegible] The base vectors for these expansions [illegible] Eqs. 9. Define the signal $S=C_iD_i$ in the [illegible] coordinates a_1 and a_2 corresponding to the [illegible] $\Delta f\cdot\Delta t$ and $f_0\cdot t_0$, and the coordinates b [illegible] responding to the components $\Delta f\cdot\Delta t$ and [illegible] $a_1=b_1$, but for simplicity of exposition we [illegible] notation). Then:

$$C_i=\sum_{k=1}^{\infty}\mu_{ki}D_k,\quad \mu_{ki}=(a_i,b_k)(i=1,2 \ldots$$

$$D_i=\sum_{k=1}^{\infty}\mu_{ik}C_k,\quad \mu_{ik}=(b_i,a_k)(i=1 \ldots$$

with the coefficients obeying the conditions [illegible]

$$\sum_{k=1}^{\infty}\mu_{ik}\mu_{jk}=(a_ia_j)=\delta_{ij},$$

$$\sum_{k=1}^{\infty}\mu_{ki}\mu_{kj}\,(b_ib_j)=\delta_{ij},$$

where δ_{ij} is Kronecker's symbol. The "[illegible]" [illegible] these vectors in the CNS can be described [illegible] By an orthogonal transformation, one obtains [illegible] *transformation* from a complex vector [illegible] complex vector space **R**, as a mapping L [illegible] **R**, such that

$$L(\alpha x+\beta y)=\alpha Lx+\beta Ly$$

identically, for all complex numbers, α and [illegible] vectors **x** and **y** in **H**. Furthermore, a linear functional on a complex vector space **H** is a complex [illegible] tion ϵ on **H** such that (Halmos, 1951) [illegible] i.e.

$$\epsilon(\mathbf{x}+\mathbf{y})=\epsilon(\mathbf{x})+\epsilon(\mathbf{y})$$

for every pair of vectors **x** and **y** in **H**, [illegible] homogeneous, i.e.,

$$\epsilon(\alpha\mathbf{x})=\alpha\epsilon(\mathbf{x})$$

for every complex number α and for every [illegible] in H. The effect of a linear functional [illegible] defined signal in the CNS would be [illegible] rotation of axes—clearly a necessity in [illegible] dimensional system conveying information.

III. FURTHER RELATIONS TO QUANTUM MECHANICS

To define a signal in $\mathcal{H}$ space, i.e., the [illegible] or *both* $\mathfrak{M}_1$ and $\mathfrak{M}_2$ subspaces, a *bilinear* [illegible] defined (Halmos, 1951, p. 12) on the [illegible] space $\mathcal{H}$ as a complex valued function ϕ on [illegible] product of $\mathcal{H}$ with itself, such that [illegible]

[illegible] then, for every $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{K}$, ϵ_y is a linear functional and φ_x is a conjugate linear functional.

To every bilinear functional, there is associated a quadratic form (Lusternik and Sobolev, 1961, p. 259) defined as follows: a bilinear functional ϕ on a complex vector space obtains a function $\hat{\phi}$, the quadratic form for each vector $\mathbf{x}$ by which $\hat{\phi}(\mathbf{x})=\hat{\phi}(\mathbf{x},\mathbf{x})$. Thus, the Hilbert-space-represented signal can be shown to be (Halmos, 1951, pp. 12–13):

$$\phi(\mathbf{x},\mathbf{y})=\hat{\phi}[\tfrac{1}{2}(\mathbf{x}+\mathbf{y})]-\hat{\phi}[\tfrac{1}{2}(\mathbf{x}-\mathbf{y})] +i\hat{\phi}[\tfrac{1}{2}(\mathbf{x}+i\mathbf{y})]-i\hat{\phi}[\tfrac{1}{2}(\mathbf{x}-i\mathbf{y})]. \quad (15)$$

From the reciprocal relations of Eqs. 1 and 6 bounding the vectors $\mathbf{x}$ and $\mathbf{y}$ of Eq. 9, $\phi(\mathbf{x},\mathbf{y})=\phi^*(\mathbf{y},\mathbf{x})$. The bilinear functional defining the signal is, therefore, symmetric over the four signal dimensions.

The representation need not be confined to an inner product space. Consider the functional space whose functions are defined on a finite interval, (a,b), and for which the Lebesgue integral exists and is finite. Then, for the equivalence classes of square integral functions (complex valued) on (a,b), we can take, as the product between the two classes F and G:

$$(F,G)=\int_b^a F(x)\overline{G(x)}dx, \quad (16)$$

[illegible]

$$\int_b^a |F(x)|^2dx \quad \text{and} \quad \int_b^a |G(x)|^2dx$$

exist and where the bar denotes complex conjugation. The space in which these integrals exist is referred to as $L_2(a,b)$ space (Dennery and Krzywicki, 1967; Bachman and Narici, 1966); the L represents the name Lebesgue and the subscript 2 indicates the integrability of the modulus of each function representing a vector belonging to $L_2(a,b)$. In the case we are interested in (as $\mathbf{y}=\bar{\mathbf{x}}$), the classes will be $\mathbf{X}$, i.e., $(\mathbf{X},\mathbf{X})$.

Suppose a vector $\mathbf{w}\epsilon L_2(a,b)$ approximated as a sequence of a finite sum of base vectors $\mathbf{e}_i\epsilon L_2{}^2(a,b)$. Let these base vectors correspond to the elements of vectors $\mathbf{x}$ and $\mathbf{y}$ of Eq. 9; then

$$\mathbf{w}=\sum_{k=1}^{\infty} w^k\mathbf{e}_i, \quad i=1,2, \quad (17)$$

where the vectors $\mathbf{e}_i$ satisfy the property

$$\mathbf{e}_1\mathbf{e}_2=\delta. \quad (18)$$

The numbers w^k in Eq. 17 are the *Fourier coefficients* of $\mathbf{w}$ with respect to the bases $\mathbf{e}_1$ and $\mathbf{e}_2$.

These Fourier coefficients satisfy the convergence condition

$$\sum_{k=1}^{\infty} |w^k|^2<\infty, \quad (19)$$

[illegible] a vector in Hilbert space (Dennery and Krzywicki 1967, p. 197). Also, a vector in Hilbert space whose components are treated as Fourier coefficients determines some vector in $L_2(a,b)$. The one-to-one correspondence between the elements of the Hilbert space and the function space $L_2(a,b)$ means that the two spaces are isomorphic.

It is possible, therefore, to represent a signal by both a quadratic form on the Hilbert space or as Fourier coefficients in isomorphic function space. Each Fourier coefficient would give a measure of the overall system in the subsystems, i.e., in this instance, two subsystems, and would correspond to the absolute value of the vectors $\mathbf{x}$ and $\mathbf{y}$. A basic principle of quantum mechanics is thus obeyed; namely, all the physically relevant information about a physical system at a given instant of time is derivable from the knowledge of the state function of the system.

IV. DISCUSSION

The analysis of this paper is based on the notion of the fundamental nature of the *distribution* of energy to forms of information. The view that the amount of energy transduced by the CNS, equated with the number of action potentials in a given period, will give a measure of (a) the amount of information in the sensory signal, or (b) the amount by which a signal needs to be increased in order to be detected, has found adherents (Fitzhugh, 1957; Barlow, 1962—in visual physiology; McGill, 1967; Green and Swets, 1966, Chap. 8—in audition and signal detection theory). By the methods presented here, this interest is now extended into an analysis of the forms in which the fundamental notion of energy can exist, i.e., how that energy is distributed. The import of this paper in-indicates further developments[3]: the succinct description of the total information content of a signal and its reduction to a set of numbers in Hilbert space implies a description of the Hamiltonian of the system. The resulting Hilbert space representation is related to the overall energy distribution of the system under consideration. Thus a development of this line of reasoning (Barrett, 1971, 1973) follows the path taken by Schrödinger with the important exception that the "Hamiltonian" describes energy distribution, rather than absolute energy amounts. There is, of course, no correspondence between the total energy in a system and the complexity of its distribution. Yet a measure of that complexity is an information measure.

[1] I am using the word "association" to imply a relation less strong than equality.

[2] For a discussion of this property see Barrett (1972c).

[3] The reviewer of this paper brought to my attention the paper by J. T. Winthrop ["Propagation of Structural Information in Optical Wave Fields," J. Opt. Soc. Am. 61, 15–30 (1971)]. Winthrop addresses the topic of structural information theory and light.

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